Sensitivity Analysis

- A few concepts
  - Impulse-Response
- What do we know how to do
  - By MP class
- Related analyses
  - Consistency, Redundancy, & Implied Equalities
- Some foundations
  - Alternative/dual systems
- Some practical considerations
  - Estimating results from partial information

An 'expert' is one who doesn't know more than you but uses slides.

Impulse-Response Queries

	Response	
Impulse	Data	Solution
Data	*Drive*	*Common*
Solution	*Inverse*	*Rate of substitution*

I never met an optimum I didn't like.
– Milton M. Gutterman

Impulse-Response Queries

	Response	
Impulse	Data	Solution
Data	*Drive*	***Common***
Solution	*Inverse*	*Rate of substitution*

- At what rate does the objective value change when I perturb some parameter? For what range is this rate constant (or same functional form)?
- At what rate does the level (or price) change when I perturb some parameter? For what range is this rate constant (or same functional form)?

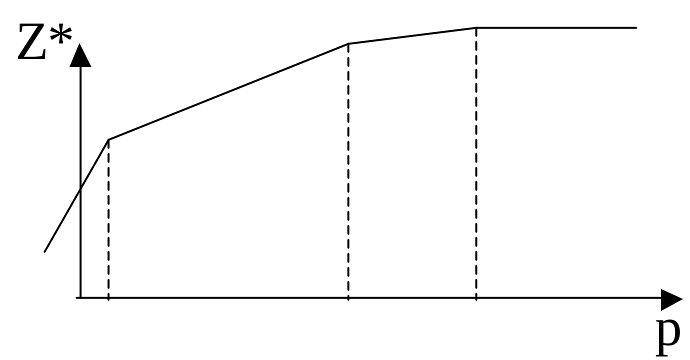
$\partial^{(\pm)} Z^*/\partial p = k$ for $p \in [P-L, P+U]$

p = parameter

P = current value of p

Z^* = optimal objective value

$[L, U]$ = range of change

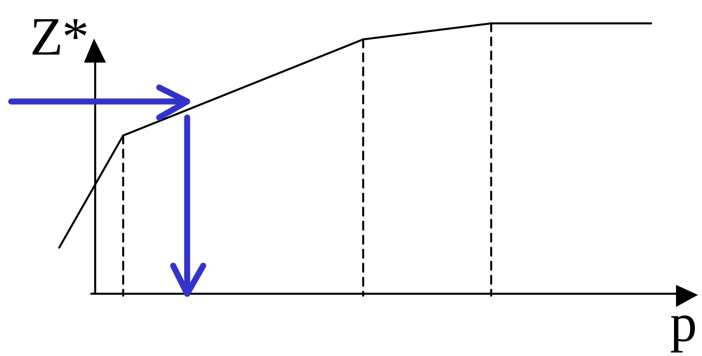


Impulse-Response Queries

	Response	
Impulse	Data	Solution
Data	*Drive*	*Common*
Solution	***Inverse***	*Rate of substitution*

- How can I change some parameter to cause a 10% decrease in the min cost?
  – e.g., decrease demand or make some inexpensive supply available
- How can I change some parameter to reach specified change in solution?
  – e.g., increase max oxygen to result in more glucose production.

$\partial^{(\pm)} Z^* / \partial p = k$ for $p \in [P-L, P+U]$

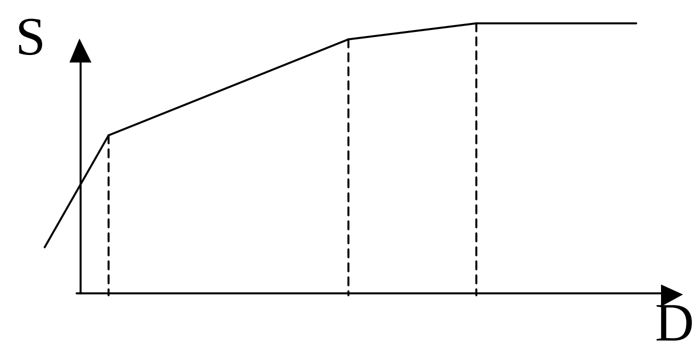


Impulse-Response Queries

	Response	
Impulse	Data	Solution
Data	***Drive***	*Common*
Solution	*Inverse*	*Rate of substitution*

- How can I change some parameter such that to remain in equilibrium, I must change another (specified) parameter?
  – e.g., decrease demand (D) and increase some (specified) supply (S):

$$\Delta S = k\Delta D$$



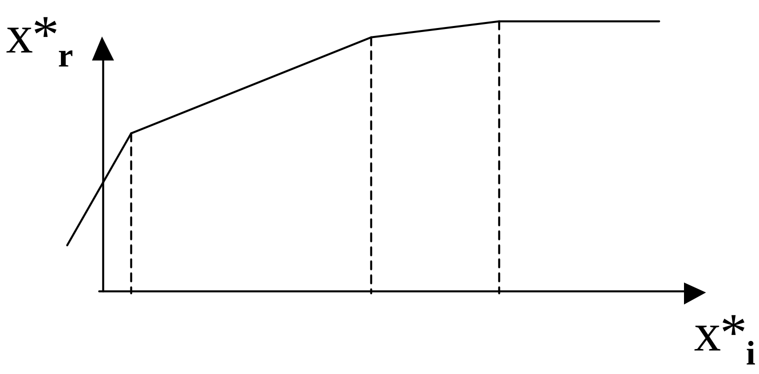
Impulse-Response Queries

	Response	
Impulse	Data	Solution
Data	*Drive*	*Common*
Solution	*Inverse*	***Rate of substitution***

- How does one solution value change if I force a change in some other?

$$\partial^{(\pm)} x^*_r / \partial x^*_i$$

– Applies to phase-plane analysis.



Simplex Method Uses this Every Iteration

		Nonbasic		
Basic	Level		x_i	
x_r	b_r		a_{ri}	
-Z			d_i	

$$x_r = b_r - a_{ri}\, x_i$$

rate of substitution

Qualitative Analysis

- Given directions of change of parameter, find directions of change of solution
- Find qualitative relations among variables (degrees of separation among metabolites or reactions)
- Find stability properties (not numbers)
- Find pathways of certain interest

Modeling is about insight, not numbers.
– Arthur M. Geoffrion

A Quick Tour of What We Know

- Linear Programming (LP)
- Nonlinear Programming (NLP)
- Integer Programming &
  Combinatorial Optimization (IP/CO)
- Mixed-Integer Linear Programming (MILP)
- Mixed-Integer Nonlinear Programming (MINLP)

The pure and simple truth is rarely pure and never simple.
– Oscar Wilde

LP

- Basic solution
  - Compatibility theory
- Interior solution
  - Optimal partition
- General case
  - Character of solution

Mostly well understood, but algorithms not perfect

Qualitative analysis strongest for network models, then Leontief

MOLP: Objective space gives important insights

NLP

- Lagrange multipliers
  - Marginal analysis with convexity; “rapid” re-optimization
- Dynamic programming
  - Inherently parametric; needs separability & low dimension
- Pooling problem (bilinear constraints)
  - Exploit geometry to overcome non-convexity
  - Raised new concept – *Essential* objects (pools/reactions)

Sometimes wrong, but never in doubt.
– Michael Evans (Economics forecaster)

IP/CO

- Generally NP-hard
  - Optimizers do not provide automatic support beyond LP
- Special focus on problem structures
  - Scheduling, TSP, covering, packing, …
- Computational logic
  - Horn clauses: if A then B (single antecedent & consequent)
- New definitions
  - Stability regions; ties with algorithm/heuristic used
- Visualization
  - Diagrammatic; Iconic; Animation

MILP

- Loses structural information
  - Preserve logic of IP part (binary variables to control fluxes)

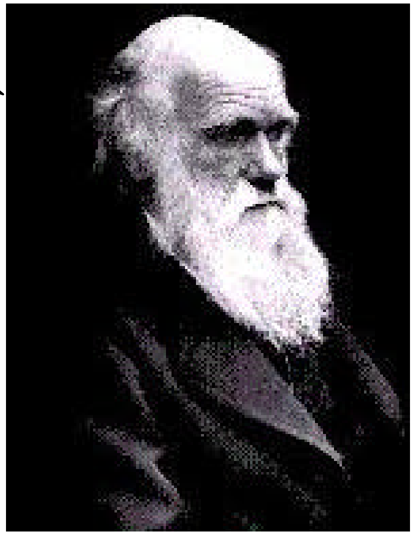
Logical	Algebraic
$x=0 \rightarrow y=0$	$x - y \geq 0$
$x=0 \rightarrow y=1$	$x + y \geq 1$
$x=1 \rightarrow y=0$	$x + y \leq 1$
$x=1 \rightarrow y=1$	$x - y \leq 0$

MINLP

- No theory; few special algorithms

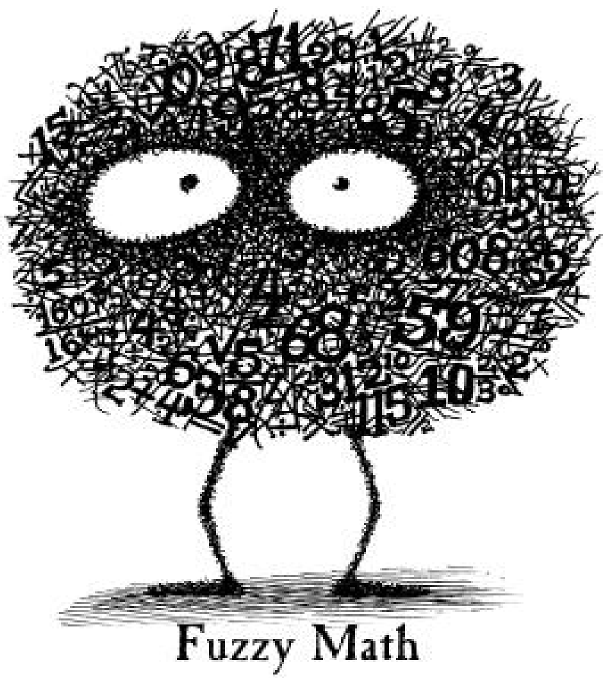
  (I. Grossman did some things for specific problems)

Every new body of discovery is mathematical in form, because there is no other guidance we can have.



Other Forms

- Multiple objectives
- Goal programs
- Fuzzy programs
- Stochastic programs
- Randomized programs
- Semi-definite programs



Summary of SA Capability

- Linear Programming (LP)
  - Lots known; All queries; Must be careful
- Nonlinear Programming (NLP)
  - Only special cases (convex quadratic; bilinear)
- Integer Programming & Combinatorial Optimization (IP/CO)
  - Hard, but some good results, using logical structure
- Mixed-Integer Linear Programming (MILP)
  - Use IP/CO methods
- Mixed-Integer Nonlinear Programming (MINLP)
  - Uncharted

Consistency, Redundancy, and Implied Equalities

System: $S = \{Ax \geq b\}$

Polyhedron: $P(S) = \{x: Ax \geq b\}$

Subsystems: $S(I) = \{A_{i\bullet}x \geq b_i \text{ for } i \in I\}$

$S_i = \{A_{k\bullet}x \geq b_k \text{ for } k \neq i\}$

Inconsistent: $P(S) = \varphi$

Redundant: $P(S_i) = P(S)$

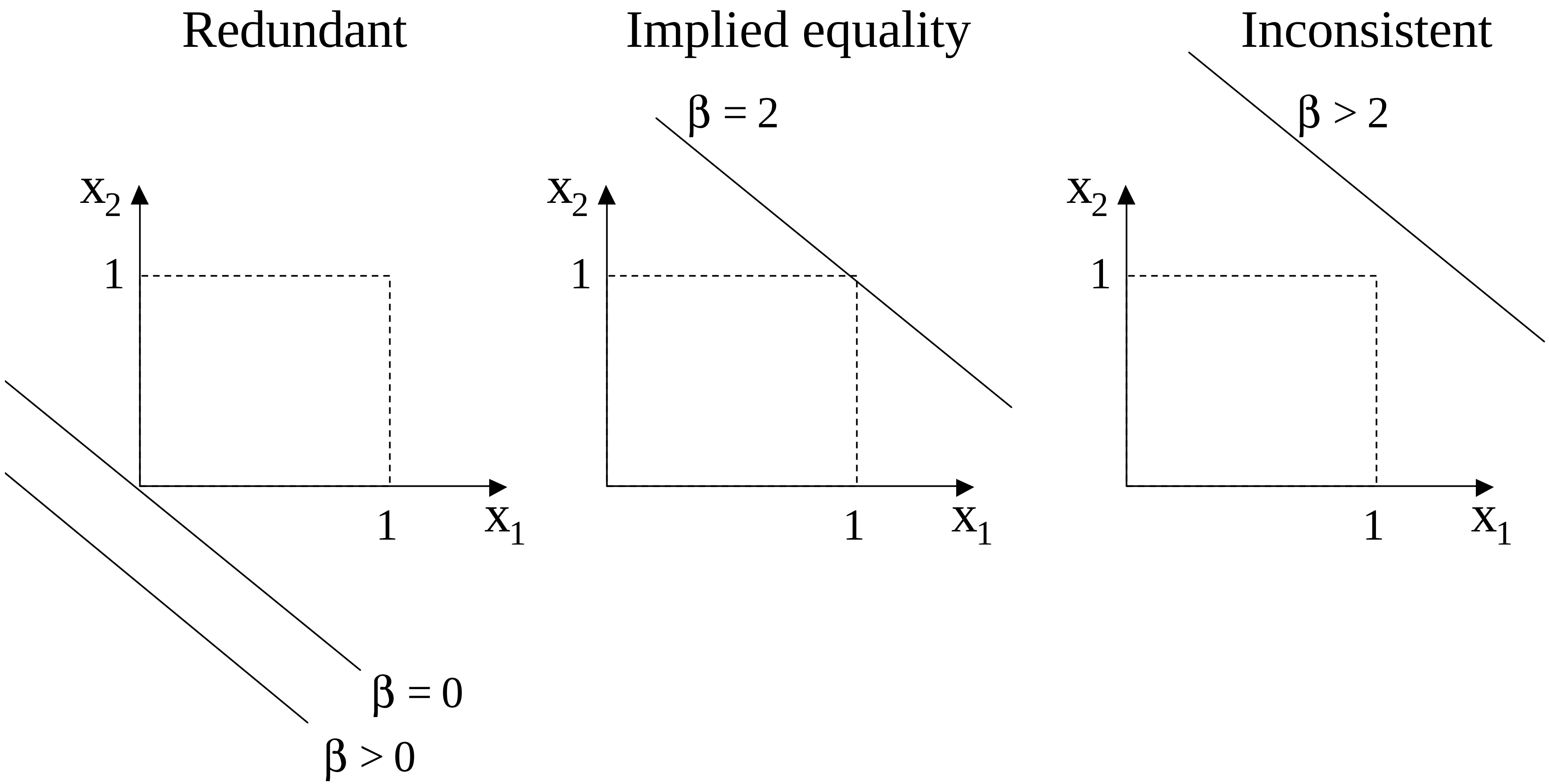
Strongly Redundant: $x \in P(S_i) \rightarrow A_{i\bullet}x > b_i$

Implied Equality: $A_{i\bullet}x = b_i \text{ for all } x \in P(S)$

A model is to an analyst as a magnifying glass is to Sherlock Holmes – it illuminates clues.

Example

$$S = \{0 \leq x_1, x_2 \leq 1, \; x_1 + x_2 \geq \beta\}$$



Foundation = Dual system

$$S^d = \{y \geq 0,\ yA = 0,\ yb \geq 0\}$$

Example

$$\begin{array}{rcrcr} x_1 & & & \geq & 0 \\ & & x_2 & \geq & 0 \\ -x_1 & & & \geq & -1 \\ & & -x_2 & \geq & -1 \\ x_1 & + & x_2 & \geq & \beta \end{array}$$

$$\begin{array}{r} y_1, y_2, y_3, y_4, y_5 \geq 0 \\ y_1 - y_3 + y_5 = 0 \\ y_2 - y_4 + y_5 = 0 \\ -y_3 - y_4 + \beta y_5 \geq 0 \end{array}$$

A study of economics usually reveals that the best time to buy anything is last year.

– Marty Allen

Certification

Property of S is true $\leftrightarrow$ S* is consistent

Property of S	S*
Redundant	$S^d \setminus \{y_i \geq 0\}$ & $\{y_i < 0\}$
Strongly redundant	$S^d \setminus \{y_i \geq 0\}$ & $\{y_i < 0, yb > 0\}$
Implied equality	S^d & $\{y_i > 0, yb = 0\}$
Inconsistent	S^d & $\{yb > 0\}$

$$y_1, y_2, y_3, y_4, y_5 \geq 0$$

$$y_1 - y_3 + y_5 = 0$$

$$y_2 - y_4 + y_5 = 0$$

$$-y_3 - y_4 + \beta y_5 \geq 0$$

Certification of Redundancy

Property of S is true $\leftrightarrow$ S* is consistent

Property of S	S*
Redundant	$S^d \backslash \{y_i \geq 0\}$ & $\{y_i < 0\}$
Strongly redundant	$S^d \backslash \{y_i \geq 0\}$ & $\{y_i < 0, yb > 0\}$
Implied equality	S^d & $\{y_i > 0, yb = 0\}$
Inconsistent	S^d & $\{yb > 0\}$

$\beta = 0$

choose $y = (1, 1, 0, 0, -1)$

$y_1, y_2, y_3, y_4, \not{y_5} \geq 0 \qquad y_5 < 0$

$y_1 - y_3 + y_5 = 0$

$y_2 - y_4 + y_5 = 0$

$-y_3 - y_4 \geq 0 \quad \rightarrow y_3 = y_4 = 0$

Certification of Strong Redundancy

Property of S is true ↔ S* is consistent

Property of S	S*
Redundant	$S^d \setminus \{y_i \geq 0\}$ & $\{y_i < 0\}$
Strongly redundant	$S^d \setminus \{y_i \geq 0\}$ & $\{y_i < 0, yb > 0\}$
Implied equality	S^d & $\{y_i > 0, yb = 0\}$
Inconsistent	S^d & $\{yb > 0\}$

$\beta < 0$

choose $y = (1, 1, 0, 0, -1)$

certifies strong redundancy
because $yb = -\beta > 0$

$y_1, y_2, y_3, y_4, \not{y_5} \geq 0 \qquad y_5 < 0$

$y_1 - y_3 + y_5 = 0$

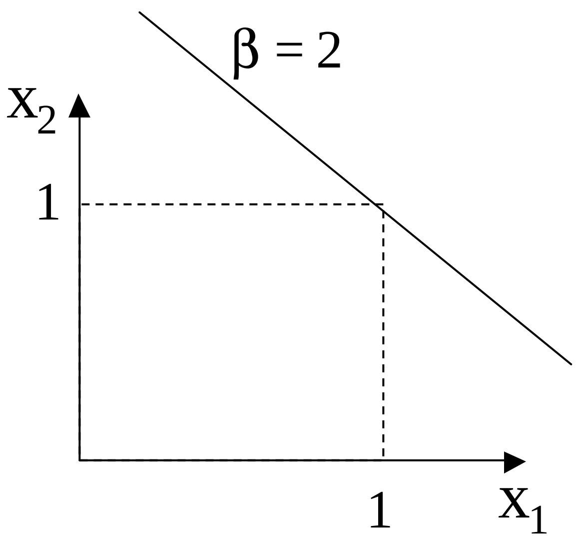
$y_2 - y_4 + y_5 = 0$

$-y_3 - y_4 + \beta y_5 \geq 0$

Certification of Implied Equality

Property of S is true $\leftrightarrow$ S* is consistent

Property of S	S*
Redundant	$S^d \setminus \{y_i \geq 0\}$ & $\{y_i < 0\}$
Strongly redundant	$S^d \setminus \{y_i \geq 0\}$ & $\{y_i < 0, yb > 0\}$
Implied equality	S^d & $\{y_i > 0, yb = 0\}$
Inconsistent	S^d & $\{yb > 0\}$



$$y_1, y_2, y_3, y_4, y_5 \geq 0$$
$$y_1 - y_3 + y_5 = 0$$
$$y_2 - y_4 + y_5 = 0$$
$$-y_3 - y_4 + \beta y_5 \geq 0$$

Certification of Implied Equality

Property of S is true $\leftrightarrow$ S* is consistent

Property of S	S*
Redundant	$S^d \setminus \{y_i \geq 0\}$ & $\{y_i < 0\}$
Strongly redundant	$S^d \setminus \{y_i \geq 0\}$ & $\{y_i < 0, yb > 0\}$
Implied equality	S^d & $\{y_i > 0, yb = 0\}$
Inconsistent	S^d & $\{yb > 0\}$

$\beta = 2$

choose y = (0, 0, 1, 1, 1)

plays no role in implied equality

$$y_1, y_2, y_3, y_4, y_5 \geq 0$$
$$y_1 - y_3 + y_5 = 0$$
$$y_2 - y_4 + y_5 = 0$$
$$- y_3 - y_4 + 2y_5 \geq 0$$

Certification of Inconsistency

Property of S is true $\leftrightarrow$ S* is consistent

Property of S	S*
Redundant	$S^d \setminus \{y_i \geq 0\}$ & $\{y_i < 0\}$
Strongly redundant	$S^d \setminus \{y_i \geq 0\}$ & $\{y_i < 0, yb > 0\}$
Implied equality	S^d & $\{y_i > 0, yb = 0\}$
Inconsistent	S^d & $\{yb > 0\}$

$\beta > 2$

choose $y = (0, 0, 1, 1, 1)$

plays no role in inconsistency

$$y_1, y_2, y_3, y_4, y_5 \geq 0$$
$$y_1 - y_3 + y_5 = 0$$
$$y_2 - y_4 + y_5 = 0$$
$$-y_3 - y_4 + \beta y_5 \geq 0$$

Certification of Inconsistency

Property of S is true $\leftrightarrow$ S* is consistent

Property of S	S*
Redundant	$S^d \backslash \{y_i \geq 0\}$ & $\{y_i < 0\}$
Strongly redundant	$S^d \backslash \{y_i \geq 0\}$ & $\{y_i < 0,\ yb > 0\}$
Implied equality	S^d & $\{y_i > 0,\ yb = 0\}$
Inconsistent	S^d & $\{yb > 0\}$

$\beta > 2$

choose $y = (0, 0, 1, 1, 1)$

certifies inconsistency
because $yb = \beta - 2 > 0$

$$y_1, y_2, y_3, y_4, y_5 \geq 0$$
$$y_1 - y_3 + y_5 = 0$$
$$y_2 - y_4 + y_5 = 0$$
$$-y_3 - y_4 + \beta y_5 \geq 0$$

Certificates Obtained by LP

$$\max yb: \; y \in P(S^{\sim}), \quad \Sigma_i \, y_i = 1$$

Normalization to have $y \neq 0$

Interior Solutions Certify All at Once with $\sigma(y)$

$$y^* \in \text{argmax}\{yb: \; yA = 0, \; y \geq 0, \Sigma_i \, y_i = 1\}$$

$\Rightarrow$ every i for which $y^*_i > 0$ is an implied equality

y^* interior $\Rightarrow \sigma(y)$ = set of ***all*** implied equalities of $Ax \geq b$