Polyhedral Computation
by Harvey Greenberg, CU-Denver

H-representation:
\{x: Ax \leq b\}

V-representation:
\text{convh}\{v_1, \ldots, v_p\}

[+ \text{convh}\{(r_1), \ldots, (r_q)\}]
Transforms $H \leftrightarrow V$

- $H \rightarrow V$ is the vertex enumeration problem (often embedded in face enumeration)
- $V \rightarrow H$ is a form of the convex hull problem (Quickhull algorithm is another form)
Convex Hulls

- Easy in 2-space (Graham scan; Jarvis march)
- $O(p^{n/2})$ in general (maybe just for simplicial polyhedra - i.e., every face is a simplex)

*Don't explain why it can't be done.*
*Discover how it can be done.*
Mo Tao (404-319 B.C.)
Enumeration of Extreme Points and Extreme Rays

- Double Description Method, based on Fourier-Motzkin elimination
  - ccd, Fukuda; dda, Padberg
- Reverse Search Algorithm, uses simplex method with systematic search over sequence of bases
  - lrs, Aris
- All implementations are very limited - about 20 variables (maybe 30 max)
- We’ll consider “column generation” to avoid explicit enumeration while satisfying some criteria
Conversion to *Standard form* to use Simplex Method

**H-representation**

\[ Ax \leq b \]

\[ \leftrightarrow Ax + s = b, \; s \geq 0 \]

\[ \leftrightarrow Au - Av + s = b, \; (u, \; v, \; s) \geq 0 \]

\[
\begin{bmatrix}
A & -A & I
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
 s
\end{bmatrix} = b
\]

**Standard form**

\[ Ax = b, \; x \geq 0, \; \text{rank}(A)=m \]

**Extreme point x**

\[ \leftrightarrow \text{Basic Feasible Solution} \]

\[ B \subseteq \{1, \ldots, n\} \text{ for which } |B|=m, \]

\[ A_B = \{A_j\}_{j \in B} \text{ are linearly independent,} \]

\[ \text{and } [A_B]^{-1}b \geq 0 \]

**Systematically enumerate all BFSs**

**Glitch:** extreme point in enlarged space \((u, \; v, \; s)\) is not necessarily extreme point in original space \((x)\)

(but pathology does not apply to bounded P).
Reverse Search

Basic idea:
1. choose $c$ so that $0 = \text{argmin}\{cx: Ax \leq 1, x \geq 0\}$ (unique)
2. build search tree and reverse the pivots in the simplex method

$c = (1, 1, \ldots, 1) \Rightarrow$ simplex pivot/search tree induced

ref.: Bremmer, et al. [1998]
Bound on Number of Extreme Points

In standard form, $|\text{ext}(P)| \leq \binom{n}{m}$

$m=1$: $a(1)x(1) + a(2)x(2) + \ldots + a(n)x(n) = b$, $x \geq 0$
To be bounded, need $a > 0$ or $a < 0$ (no mixed signs)
To be full dimensional, need $b \neq 0$

$\text{ext}(P) = \{(b/a(1), 0, \ldots, 0), (0, b/a(2), 0, \ldots, 0), \ldots, (0, \ldots, 0, b/a(n))\}$

So, $|\text{ext}(P)| = n$ (i.e., bound is tight for $m=1$)

$a(1)x(1) + a(2)x(2) + x(3) = b$
$(a(1)x(1) + a(2)x(2) \leq b)
Factorials grow exponentially

In standard form, \(|\text{ext}(\mathcal{P})| \leq \binom{n}{m} = \frac{n!}{m!(n-m)!} \)

Stirling’s approximation: \(n! \approx \sqrt{2\pi n^{n+\frac{1}{2}} e^{-n}}\)

Low end - \(m \leq 10\% \ n\): \(\binom{n}{m} \approx c \frac{(1+r)^n}{\sqrt{n}} : r < \frac{1}{2}\)

High end - \(m \approx n/2\): \(\binom{n}{m} = \frac{2^{n+1}}{\sqrt{2\pi n}}\)
Numbers are huge

In standard form, \( |\text{ext}(P)| \leq \binom{n}{m} = \frac{n!}{m!(n-m)!} \)

\[
m \text{ fixed: } \binom{n}{m} \approx \frac{n^{n+\frac{1}{2}}}{m!(n-m)^{n-m+\frac{1}{2}} e^m} = O(n^m)
\]

<table>
<thead>
<tr>
<th>(n)</th>
<th>(\binom{n}{5})</th>
<th>(\binom{n}{6})</th>
<th>(\binom{n}{7})</th>
<th>(\binom{n}{8})</th>
<th>(\binom{n}{9})</th>
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<tbody>
<tr>
<td>20</td>
<td>(1.6 \times 10^4)</td>
<td>(3.8 \times 10^4)</td>
<td>(7.8 \times 10^4)</td>
<td>(1.3 \times 10^5)</td>
<td>(1.7 \times 10^5)</td>
</tr>
<tr>
<td>30</td>
<td>(1.4 \times 10^5)</td>
<td>(5.9 \times 10^5)</td>
<td>(2.0 \times 10^6)</td>
<td>(5.9 \times 10^6)</td>
<td>(1.4 \times 10^7)</td>
</tr>
<tr>
<td>40</td>
<td>(6.6 \times 10^5)</td>
<td>(3.8 \times 10^6)</td>
<td>(1.9 \times 10^7)</td>
<td>(7.7 \times 10^7)</td>
<td>(2.7 \times 10^8)</td>
</tr>
<tr>
<td>50</td>
<td>(2.1 \times 10^6)</td>
<td>(1.6 \times 10^7)</td>
<td>(1.0 \times 10^8)</td>
<td>(5.4 \times 10^8)</td>
<td>(2.5 \times 10^9)</td>
</tr>
<tr>
<td>100</td>
<td>(7.5 \times 10^7)</td>
<td>(1.2 \times 10^9)</td>
<td>(1.6 \times 10^{10})</td>
<td>(1.9 \times 10^{11})</td>
<td>(1.9 \times 10^{12})</td>
</tr>
</tbody>
</table>

| 200  | \(7.5 \times 10^8\) | \(1.2 \times 10^{10}\) | \(1.6 \times 10^{10}\) | \(1.9 \times 10^{11}\) | \(1.9 \times 10^{12}\) |
| 200  | \(2.5 \times 10^9\) | \(8.2 \times 10^{11}\) | \(2.3 \times 10^{13}\) | \(5.5 \times 10^{14}\) | \(1.2 \times 10^{15}\) |

order of magnitude not reliable at low end
Numbers are huge

In standard form, \( |\text{ext}(P)| \leq \binom{n}{m} = \frac{n!}{m!(n-m)!} \)

Low end - \( m \leq 10\% n \):
\[
\binom{n}{m} \approx c \frac{(1+r)^n}{\sqrt{n}} : r < \frac{1}{2}
\]

<table>
<thead>
<tr>
<th>n</th>
<th>m = 10% n</th>
<th>( \binom{n}{m} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>2</td>
<td>190</td>
</tr>
<tr>
<td>30</td>
<td>3</td>
<td>4,060</td>
</tr>
<tr>
<td>40</td>
<td>4</td>
<td>91,390</td>
</tr>
<tr>
<td>50</td>
<td>5</td>
<td>( 2.11 \times 10^6 )</td>
</tr>
<tr>
<td>100</td>
<td>10</td>
<td>( 1.73 \times 10^{13} )</td>
</tr>
<tr>
<td>200</td>
<td>20</td>
<td>( 1.61 \times 10^{27} )</td>
</tr>
</tbody>
</table>

factoid: postulated age of universe = \( 10^{17} \) seconds
Numbers are huge

In standard form, \(|\text{ext}(P)| \leq \binom{n}{m} = \frac{n!}{m!(n-m)!}\)

Latest \(E. coli\) network: \(\binom{931}{626} \approx 1.5 \times 10^{254}\)

High end - \(m \approx n/2\): \(\binom{n}{m} = \frac{2^{n+1}}{\sqrt{2\pi n}}\)

<table>
<thead>
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<tr>
<td>20</td>
<td>1.8 \times 10^5</td>
</tr>
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<td>1.4 \times 10^{11}</td>
</tr>
<tr>
<td>50</td>
<td>1.3 \times 10^{14}</td>
</tr>
<tr>
<td>100</td>
<td>1.0 \times 10^{29}</td>
</tr>
<tr>
<td>200</td>
<td>9.0 \times 10^{58}</td>
</tr>
</tbody>
</table>
Bounds are not Counts

What portion of the possible extreme points are in fact present? How can we find out?

I've been so thoroughly trained that I don't even have to think before I speak.
Finding extreme points in $V$-representation is easy problem

Q: Is $v_s$ an extreme point of $\text{convh}\{v_1, \ldots, v_p\}$?

A: No iff $0 = \min\{w_s: w \geq 0, \sum_i w_i = 1, v_s = \sum_i w_i v_i\}$.

$w = e_s$ is feasible

\[= (0, \ldots, 0, 1, 0, \ldots, 0)\]

\[\uparrow\]

$s$ coordinate

$w_s = 0$ means we have $v_s = \sum_{i \neq s} w_i v_i$

$\leftrightarrow v_s \in \text{convh}\{v_1, \ldots, v_{s-1}, v_{s+1}, \ldots, v_p\}$
Inclusion questions

Is \( x \) in \( P \)?
- **H-representation**: Compute \( Ax \) and compare with \( b \)
- **V-representation**: Is \( x \) in \( \text{convh}\{v_j\} \)?
  
  \[
  \min w_0 : x = \sum_j w_j v_j + w_0 x : \ w \geq 0, \ \sum_{j=0} w_j = 1 \\
  = 0 \iff \text{yes}
  \]

- **Extension**: If no, give separating hyperplane
  
  \[
  \min \sum_j w_j : \ w \geq x-u, \ w \geq u-x, u \in P
  \]

  \[
  w_j = |x_j - u_j| \text{ at min}
  \]

  ‘\( u \) in \( P \)’ easy for H-or V-representation

\[
a = (x-u); \ H=\{v : av=a(x+u)/2\}
\]
Inclusion questions

Is $P \cap Q = \varnothing$? Let $P = \text{convh}\{v_j\}$ and $Q = \{x: Ax \leq b\}$
- $\min y_0: x = \sum_j y_j v_j + y_0 x$: $y \geq 0$, $\sum_{j=0} y_j = 1$, $Ax \leq b$
- $= 0 \iff$ no ($x$ is in $P \cap Q$)

- Extension: If so, give separating hyperplane

$$\min \sum_j w_j: w \geq x - u, \ w \geq u - x, \ u \text{ in } P, \ x \text{ in } Q$$

$$w_j = |x_j - u_j| \text{ at min}$$

simply linear constraints

$$a = (x - u); \ H = \{v: av = a(x + u)/2\}$$
Volume Computation

- Exact formula
- Simplicial subdivision
- Monte Carlo
- Heuristics

😊 I thought this was new - hadn’t found in literature; thanks to Steve Bell for pointing to Lovász’s paper (added to refs)
Exact Formula for Polytope

Assume $P=\{x: Ax \leq b\}$ is simple
− i.e., $|\{i: A(i, \bullet)x = b(i)\}| = n$
for all $x \in \text{ext}(P)$

Assume $0 \in \text{ext}(P)$ and $P \subset \mathbb{R}^+$

Let $f(x) = c'x + d$ such that $f$ is non-constant on each edge of $P$

for each $v \in \text{ext}(P)$,

$$S = \{i: A(i, \bullet)x = b(i)\}$$

$$D = |\det(A_{S})|$$

(can update with pivots)

$$w = [A_{S}]^{-1}c$$

(i.e., $c = w_{1}A_{j_{1}} + \ldots + w_{n}A_{j_{n}}$

$$\text{vol}(P) = \sum_{v \in \text{ext}(P)} N(v)$$

$$N(v) = \frac{f(v)^{n}}{n!Dw_{1} \ldots w_{n}}$$
Example

\[ \begin{align*}
-x_1 & \leq 0 \\
-x_2 & \leq 0 \\
x_1 & \leq 2 \\
x_2 & \leq 2 \\
x_1 + x_2 & \leq 3
\end{align*} \]

\[ \text{ext}(P) = \{(0, 0), (2, 0), (2, 1), (1, 2), (0, 2)\} \]

\[ c' = (1, -1), \quad d=0 \implies f(x) = x_1 - x_2 \]

\[ v_1 = (0, 0): \quad A_{s_1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad D=1, \quad w=(-1, 1)' \implies N(v_1) = 0 \]

\[ v_2 = (2, 0): \quad A_{s_2} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad D=1, \quad w=(1, 1)' \implies N(v_2) = \frac{2^2}{2! 1 \cdot 1} = 2 \]

\[ v_3 = (2, 1): \quad A_{s_3} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad D=1, \quad w=(2, -1)' \implies N(v_3) = \frac{1^2}{2! 2 \cdot -1} = -\frac{1}{4} \]
Example (con’t)

\[ \begin{align*}
-x_1 &\leq 0 \\
-x_2 &\leq 0 \\
x_1 &\leq 2 \\
x_2 &\leq 2 \\
x_1 + x_2 &\leq 3
\end{align*} \]

\[ \text{ext}(P) = \{(0, 0), (2, 0), (2, 1), (1, 2), (0, 2)\} \]

\[ \text{v}_4 = (1, 2): \quad A_{S_4} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad D=1, \quad w=( -2, 1)' \Rightarrow N(v_4) = \frac{1^2}{2! -2\cdot1} = -\frac{1}{4} \]

\[ \text{v}_5 = (0, 2): \quad A_{S_5} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D=1, \quad w=(-1,-1)' \Rightarrow N(v_5) = \frac{2^2}{2! -1\cdot-1} = 2 \]

\[ \text{vol}(P) = 0 + 2 – \frac{1}{4} – \frac{1}{4} + 2 = 3\frac{1}{2} \]

By inspection

\[ \text{vol}(P) = \text{vol}(\bullet ) – \text{vol}(\blacklozenge) = 4 – \frac{1}{2} = 3\frac{1}{2} \]
Simplicial Subdivision

Volume of one simplex:

$$\text{vol}(\text{convh}\{0, v_1, \ldots, v_n\}) = \frac{\det[v_1 \ldots v_n]}{n!}$$

$v_1 = (2, 0)^t$

$v_2 = (0, 1)^t$

$$\text{vol} = \frac{1}{2} \det \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

$P = \bigcup S_i \text{ s.t. } \text{int}(S_i) \cap \text{int}(S_j) = \emptyset \text{ for } i \neq j$

$$\Rightarrow \text{vol}(P) = \sum_i \text{vol}(S_i)$$
Other subdivisions

We can decompose $P = P_1 \cup P_2 \cup \ldots \cup P_k$ such that
\[
\text{int}(P_i) \cap \text{int}(P_j) = \emptyset \quad \text{for } i \neq j
\]
so $\text{vol}(P) = \text{vol}(P_1) + \text{vol}(P_2) + \ldots + \text{vol}(P_k)$

- We know we can do it with simplexes, but we might not be able to “tile” $P$ with other shapes (like squares).
- We could approximate $\text{vol}(P)$ with inner and outer approximations that are easy to compute.

*Mathematicians are like Frenchmen: whenever you say something to them, they translate it into their own language, and at once it is something entirely different.*

J.W.v. Goethe
Approximations

- **Inner** - find $\text{vol}(Q)$ for $Q \subseteq P$
  \[ \max \sum_j \log x(j): \; x \in P, \; x > 0 \]
  - easy convex program with linear constraints

- **Outer** - find $\text{vol}(Q)$ for $Q \supseteq P$
  \[ \max x(j): \; x \in P \]
  - $n$ LPs (or $2n$ LPs if $\min x(j)$ could be $> 0$)
Monte Carlo

Solve $L_j = \min\{x_j : Ax \leq b\}$ and $U_j = \max\{x_j : Ax \leq b\}$

If $L_j = U_j$ for some $j$, eliminate $x_j$.

Now $L < U$ and assume $P$ has full dimension.

Choose random number sequence and choose associated $x$ in $[L, U]$. Let $y(k) = \#$ times $Ax \leq b$ in $k$ trials. Then,

$$\text{vol}(P) = \text{vol}([L, U]) \lim_{k \to \infty} \frac{y(k)}{k}$$

$$= \prod_{j=1}^{n} (U_j - L_j) \ p$$

Extends using any $Q$ for which $\text{vol}(Q)$ is known, $P \subseteq Q$, and we can map random number into a point in $Q$. 

$\text{vol}(Q) = 32$
Choosing random points

for j=1:n
    r = pseudo random value in (0, 1)
    x_j = L_j + r × (U_j − L_j)
end

vol(P) est. = (4/7) × 32 = 18.29

Turn to MATLAB code demo

Count = 0;
for k=1:MaxIter
    x = rand(n,1).*U;
    if A*x <= b, Count = Count+1;
end
end
p = Count/MaxIter

* Sharon Wiback had already implemented a version of this
Choosing tighter parallelepiped enclosure

\[ Q = \text{parallelepiped} = \{ \sum_j y_j v_j : 0 \leq y_j \leq 1 \}, \]
where \( \{v_j\} \) are linearly independent

\[
\text{vol}(Q) = |\det[v_1 \ldots v_n]| 
\]

for \( j=1:n \)
\[
y_j = \text{pseudo random value in } (0, 1) 
\]
\[
x_j = y_j \times v_j 
\]
end

output: \( x = \sum_j y_j v_j \) for \( y \sim \text{IIDU}(0, 1) \)

\[
\text{vol}(Q) = \det \begin{pmatrix} 6.4 & 3.2 \\ 2.5 & -2.2 \end{pmatrix} = 22.08 
\]

\[
\text{vol}(P) \text{ est.} = (6/7) \times 22.08 = 18.93 
\]
Advantage of tighter Q

Given P ⊂ Q ⊂ Q’

Claim: Convergence of Monte Carlo is better using Q than Q’. 

\[ \text{var(Count/k)} = p(1-p)/k, \quad \text{where} \quad p = \frac{\text{vol(P)}}{\text{vol(Q)}}, \]

so p near 1 or 0 is better than p near \( \frac{1}{2} \).

Near 0, however, has other problems with sample size.

Probabilists use the ratio \( \text{var:mean} \) as measure of convergence

→ minimizing \( 1-p \) is best.

i.e., Make vol(Q) as close to vol(P) as possible.

Computational note: vol(Q) and vol(P) could be very large, so either scale or estimate \( \log \text{vol}(P) = \log \text{Count/k} + \log \text{vol}(Q) \).
Other areas of potential value

• Volonoi diagrams & Delaunay tessellations
• Sampling techniques
• Comparing polyhedra seems to be limited to 3D

*Even if you're on the right track, you'll get run over if you just sit there.*
Will Rogers
References


